Hamiltonian systems with dissipation

Marius Buliga@imar.ro

IMAR

chorasimilarity.wordpress.com

Lagrangian mechanics

- Lagrangian:

$$L(t, q, \dot{q}) = \hat{T}(\dot{q}) - \mathcal{E}(t, q)$$

- evolution equation:

$$D_q L(t, q, \dot{q}) - \frac{d}{dt} D_{\dot{q}} L(t, q, \dot{q}) = 0$$

Lagrangian mechanics

- Lagrangian, example:

$$L(t, q, \dot{q}) = m\frac{\|\dot{q}\|^2}{2} - (E(q) - f(t)q)$$

- evolution equation:

$$-D_q E(q) + f(t) - \frac{d}{dt}(m\dot{q}) = 0$$

Hamiltonian mechanics

- Hamiltonian:

$$H(t, q, p) = T(p) + \mathcal{E}(t, q)$$

- evolution equation:

$$\left\{ \begin{array}{rcl} -\dot{p} & = & D_q H(t, q, p) \\ \dot{q} & = & D_p H(t, q, p) \end{array} \right.$$

Hamiltonian mechanics

- Hamiltonian, example:

$$H(t, q, p) = \frac{1}{2m}\|p\|^2 + (E(q) - f(t)q)$$

- evolution equation:

$$\left\{ \begin{array}{rcl} -\dot{p} & = & D_q E(q) - f(t) \\ \dot{q} & = & \frac{1}{m} p \end{array} \right.$$

Hamiltonian mechanics

- evolution equation:

$$\begin{cases} -\dot{p} & = & D_q H(t,q,p) \\ \dot{q} & = & D_p H(t,q,p) \end{cases} \qquad (1)$$

- Let $J(q,p) = (-p,q)$, then (1) is equivalent with:

$$\frac{d}{dt}(q,p) + J\,D_{(q,p)}H(t,q,p) = 0$$

Hamiltonian mechanics

- Let $z = (q, p)$. Define the Hamiltonian vector field associated to H as

$$X\,H(t,z) \,=\, -J\,D_z H(t,z)$$

- Then (1) is equivalent with:

$$\dot{z} \,-\, X\,H(t,z) \,=\, 0 \qquad (2)$$

- Interesting fact: the evolution is conservative, i.e. non-dissipative.

$$D_z H(t,z)\,\dot{z} \,=\, 0$$

Introducing dissipation

- Two functions: the Hamiltonian $H = H(t,z)$ and a convex dissipation potential $\phi = \phi(z, \dot{z})$.
- decompose the evolution into conservative and dissipative parts:

$$\dot{z}(t) \,=\, \dot{z}_C(t) + \dot{z}_D(t) \quad , \quad \dot{z}_D \,=\, \dot{z} \,-\, X\,H(t,z) \qquad (3)$$

- Evolution equation:

$$\dot{z}_D \,\in\, \partial^{\omega}\left(\phi(z,\cdot)\right)(\dot{z}) \qquad (4)$$

Introducing dissipation

- Evolution equation:

$$\dot{z}_D \in \partial^\omega \left(\phi(z,\cdot)\right)(\dot{z})$$

Comments:

- Introduced in
  - M. Buliga, Hamiltonian inclusions with convex dissipation with a view towards applications, Mathematics and its Applications 1, 2 (2009), 228-251, arXiv:0810.1429
- the notation $\partial^\omega \phi$ means the symplectic subgradient of ϕ.

Related

- In the Lagrangian formalism this can be traced back to Rayleigh and Kelvin
  - cf. Thomson and Tait L. Thomson, P.G. Tait, Principles of Mechanics and Dynamics, (1912)
- autonomous Hamiltonian systems with a Rayleigh dissipation function added:
  - A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, T.S. Ratiu, Dissipation induced instabilities, *Ann. de l'Institut Henri Poincaré Analyse non linéaire*, **11** (1994), 1, 37-90

Related

- other proposals for generalizations of lagrangian or hamiltonian mechanics by multivariate analysis:
  - J.-P. Aubin, A. Cellina, J. Nohel, Monotone trajectories of multivalued dynamical systems, *Annali di Matematica Pura ed Appl.*, **115** (1977), 99-117
  - F. Clarke, Necessary Conditions in Dynamic Optimization, Mem. AMS **816**, no. 173 (2005)
  - R.T. Rockafellar, Generalized Hamiltonian equations for convex problems of Lagrange, *Pacific J. of Math.*, **33** (1970), no. 2, 411-427
- may be related to Aubin's viability theory, but not clear how, especially in the case of 1-homogeneous ϕ.
- it looks like a dynamic version of Mielke theory of quasistatic rate-independent processes, in this case of 1-homogeneous ϕ
  - A. Mielke, Evolution in rate-independent systems (Ch. 6). In C. Dafermos, E. Feireisl, eds., *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, 461-559, Elsevier B.V., Amsterdam, 2005

Introducing dissipation

- Evolution equation:

$$\dot{z}_D \in \partial^{\omega} \left(\phi(z, \cdot)\right) (\dot{z})$$

- work in progress with Géry de Saxcé on a more elegant approach called "the symplectic Brezis-Ekeland-Nayroles principle",
  - H. Brezis and I. Ekeland, Un principe variationnel associé à certaines équations paraboliques. I. Le cas indépendant du temps, II. Le cas dépendant du temps. C. R. Acad. Sci. Paris, Série A-B, 282, 971–974, and 1197–1198, 1976
  - B. Nayroles, Deux théorèmes de minimum pour certains systèmes dissipatifs, C. R. Acad. Sci. Paris Série A-B, 282, A1035–A1038, 1976
- based on the symplectic Fenchel inequality

$$\psi(z) + \psi^{*,\omega}(z') \geq \omega(z, z')$$

Consequences

- Evolution equation:

$$\dot{z}_D \in \partial^{\omega} \left(\phi(z, \cdot) \right) (\dot{z})$$

- the evolution equation is equivalent with

$$J\,\dot{z} - D_z H(t, z) \in \partial \left(\phi(z, \cdot) \right) (\dot{z})$$

i.e. for any u

$$\phi(z, \dot{z} + u) \geq \phi(z, \dot{z}) + \langle J\dot{z} - D_z H, u \rangle$$

- suppose that $\phi(z, 0) = 0$ and $\phi(z, \dot{z}) \geq 0$, then take $u = -\dot{z}$

$$0 = \phi(z, 0) \geq \phi(z, \dot{z}) + \langle J\dot{z} - D_z H, -\dot{z} \rangle = D_z H(t, z)\dot{z}$$

therefore the system dissipates!

$$D_z H(t, z)\dot{z} \leq 0$$

Particular cases

- Evolution equation:

$$\dot{z}_D \in \partial^{\omega} \left(\phi(z, \cdot) \right) (\dot{z})$$

- $\phi = 0$, of course this is Hamiltonian mechanics.

- $\phi(z, \dot{z}) = \| \dot{z} \|^2$, this is Rayleigh dissipation

- $\phi(z, \dot{z}) = \| \dot{z} \|$, this gives a dynamic version of Mielke theory of quasistatic rate-independent processes

- $H = 0$ gives an interesting fixed point problem:

$$\dot{z} \in \partial^{\omega} \left(\phi(z, \cdot) \right) (\dot{z})$$

Variational approximation of a Mumford-Shah energy

- M. Focardi, On the variational approximation of free-discontinuity problems in the vectorial case, *Mathematical Models and Methods in Applied Sciences (M3AS)*, **11** (2001), 4, 663-684

$$E_c(\mathbf{u}, d) \; = \; \int_\Omega \left\{ \phi(d) \, w(\nabla \mathbf{u}) \, + \, \frac{1}{2} \gamma \, c \; \mid \nabla d \mid^2 \, + \frac{\gamma}{2c} \, d^2 \right\} \qquad (5)$$

is a variational approximation, as $c \to 0$ of the Mumford-Shah energy:

$$E(\mathbf{u}, S) \; = \; \int_\Omega w(\nabla \mathbf{u}) \, \mathrm{d}x \, + \, \gamma \, \mathcal{H}^2(S) \qquad . \qquad (6)$$

On the Mumford-Shah Energy

- Starting with the foundational papers of Mumford, Shah / De Giorgi, Ambrosio / Ambrosio / the development of models of quasistatic brittle fracture based on Mumford-Shah functionals continues with Francfort, Marigo / Mielke / Dal Maso, Francfort, Toader / Buliga.
- All these models are based on a technique of time discretization followed by a sequence of incremental minimization problems. These models are either seen as applications
  - of De Giorgi method of energy minimizing movements,
  - or in the frame of the theory of Mielke of rate-independent evolutionary processes.

$$E(\mathbf{u}, S) \ = \ \int_{\Omega} w(\nabla \mathbf{u}) \, \mathrm{d}x + \gamma \, \mathcal{H}^2(S) \qquad .$$

A good energy for damage

- Let's take seriously this functional as a good energy for a brittle damage model:

$$E_c(\mathbf{u}, d) \ = \ \int_\Omega \left\{ \phi(d)\, w(\nabla \mathbf{u}) \ + \ \frac{1}{2}\gamma\, c \ \mid \nabla d \mid^2 \ + \frac{\gamma}{2c}\, d^2 \right\} \qquad (7)$$

  because $d \in [0, 1]$ is good for a damage variable.
- here ϕ is a decreasing function from $[0, 1]$ to $[0, 1]$ such that $\phi(0) = 1$ and $\phi(1) = 0$
- $\phi(d)\, w(\nabla \mathbf{u})$ is the damaged elastic energy density
- $\frac{1}{2}\gamma\, c \ \mid \nabla d \mid^2 \ + \frac{\gamma}{2c}\, d^2$ is a **nonlocal** damage energy density, compatible with
  - H. Stumpf, K. Hackl, Micromechanical concept for the analysis of damage evolution in thermo-viscoelastic and quasi-brittle materials, *Int. J. of Solids and Structures*, **40** (2003), 1567-1584

The Hamiltonian

- $q = (\mathbf{u}, d)$ and $p = (\mathbf{p}, y)$
- Let us define the the Hamiltonian as:

$$H(t, q, p) = E_c(\mathbf{u}, d) + T(\mathbf{p}, y) - \langle l(t), \mathbf{u} \rangle \tag{8}$$

- where the kinetic energy is

$$T(\mathbf{p}, y) = \int_\Omega \left[\frac{1}{2} \gamma \, c \mid y \mid^2 + \frac{1}{2\rho} \|\mathbf{p}\|^2 \right]$$

- $\gamma \, c$ is a microinertia scalar, cf. Stumpf and Hackl

The dissipation potential

- the dissipation potential is inspired from
  - A. Mielke, T. Roubíček, Rate-independent damage processes in nonlinear elasticity, *Mathematical Models and Methods in Applied Sciences (M3AS)*, **16** (2006), 2, 177-209

$$\phi = \phi(\dot{d}) = \int_\Omega \left[\chi_{[0,1]}(d) + \chi_{[0,+\infty)}(\dot{d}) + \beta \mid \dot{d} \mid \right] \tag{9}$$

- it depends only on the "dissipative variable" $\dot{d}$.

The equations of the model

- the equations coming from the "non-dissipative variables" $(\mathbf{u}, \mathbf{p})$ are the usual balance equations and boundary conditions, like

$$div\ (\phi(d)\, Dw(\nabla \mathbf{u})) + f(t) = \dot{\mathbf{p}}$$

- because $\phi = \phi(\dot{d})$ we get

$$\mathbf{p} = \rho \dot{\mathbf{u}}$$

$$\dot{d} = \gamma\, c\, y$$

and ...

The equations of the model

- ... and for all $\hat{d}$, such that $\hat{d}(x) + \dot{d} \geq 0$

$$\beta \int_\Omega \left[\mid \dot{d} + \hat{d} \mid - \mid \dot{d} \mid \right] \geq \tag{10}$$

$$\geq - \int_\Omega \left[(\frac{\gamma}{c} d + \phi'(d) w(\nabla \mathbf{u}) + \dot{y}) \hat{d} + \gamma \, c \nabla d \, \nabla \hat{d} \right] \quad .$$

The equations of the model

Eventually we get the following constitutive law of brittle damage evolution:

$$- \left(\ddot{d} + \gamma^2 d + \gamma c \, \phi'(d) w(\nabla \mathbf{u}) - \gamma^2 c^2 \Delta d \right) \in \begin{cases} \gamma c \beta & , \quad \dot{d} > 0 \\ (-\infty, \gamma c \beta] & , \quad \dot{d} = 0 \end{cases}$$

What else?

The same can be done (work in progress with Géry de Saxcé) for:

- plasticity
- friction
- ... you name it and we may try to do it!